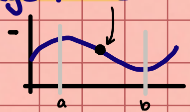
Calc 2 Final Formula

Unit 1 - 5.5, 5.6, 6.1 - 6.4

U-Substution - $\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$
- where $u = g(x)$ and $du = g'(x)$

Area Under Curve - $\int_a^b f(x)\,dx$
- under the x axis makes sign neg
- use symetry

Average Value - $\frac{1}{b-a}\int_a^b f(x)\,dx$



Disk Method - $\pi\int_a^b R(x)^2\,dx$
- distance between function and axis

Shell Method - $\int_a^b 2\pi x h\,dx$
- $r = x$; usually leave it like that
- h = height; usually the equation
- bounds are like width.

SA of Straight Line - $\int_a^b 2\pi\, \text{radius}_{avg}\, \text{length}$

Area Between Curves - $\int_a^b [f(x) - g(x)]\,dx$
- if not given bounds set equations equal
- for horizontal; solve in terms of y and bounds also have to be in y.

Volume Using Cross Sections - $V = \int_a^b A(x)\,dx$
- Cylinder → Area of circle
- Square Pyramid → Area of square

Washer Method - $\int_a^b \pi R^2 - \pi r^2$
- R = big radius; r = small radius. One radius is usually constant but the other depends on x

Arc Length - $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$

SA Curved Line - $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$

Unit 2 - 8.2 - 8.4, 8.8, 10.1

Integration by Parts - $\int u\,dv = uv - \int v\,du$
- u to du; derivative: Choose u using **L**og **I**nverse Trig **A**lgebraic **T**rig **E**xponential
- dv to v; integral: Most likely the more complicated f(x).

Trig Integrals - $\int \sin^m x \cos^n x\,dx$

Products and (some) Quotients of Trig Functions

For $\int \sin^n x \cos^m x\,dx$ we have the following:

1. ***n* odd.** Strip 1 sine out and convert rest to cosines using $\sin^2 x = 1 - \cos^2 x$, then use the substitution $u = \cos x$.
2. ***m* odd.** Strip 1 cosine out and convert rest to sines using $\cos^2 x = 1 - \sin^2 x$, then use the substitution $u = \sin x$.
3. ***n* and *m* both odd.** Use either 1. or 2.
4. ***n* and *m* both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n x \sec^m x\,dx$ we have the following:

1. ***n* odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \sec x$.
2. ***m* even.** Strip 2 secants out and convert rest to tangents using $\sec^2 x = 1 + \tan^2 x$, then use the substitution $u = \tan x$.
3. ***n* odd and *m* even.** Use either 1. or 2.
4. ***n* even and *m* odd.** Each integral will be dealt with differently.

Trig Formulas: $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Ex. $\int \tan^3 x \sec^5 x\,dx$

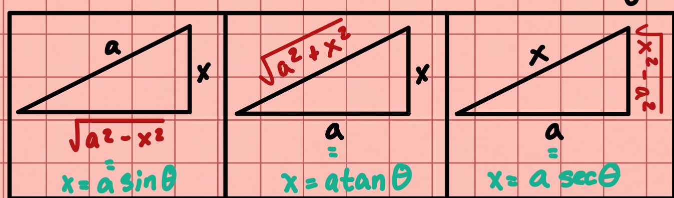
$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x \tan x \sec x\,dx$$
$$= \int (\sec^2 x - 1)\sec^4 x \tan x \sec x\,dx$$
$$= \int (u^2 - 1)u^4\,du \qquad (u = \sec x)$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + c$$

Ex. $\int \frac{\sin^5 x}{\cos^3 x}\,dx$

$$\int \frac{\sin^5 x}{\cos^3 x}\,dx = \int \frac{\sin^4 x \sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2 \sin x}{\cos^3 x}\,dx$$
$$= \int \frac{(1 - \cos^2 x)^2 \sin x}{\cos^3 x}\,dx \qquad (u = \cos x)$$
$$= -\int \frac{(1 - u^2)^2}{u^3}\,du = -\int \frac{1 - 2u^2 + u^4}{u^3}\,du$$
$$= \tfrac{1}{2}\sec^2 x + 2\ln|\cos x| - \tfrac{1}{2}\cos^2 x + c$$

- For products of trig f(x)
  - $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$
  - $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$
  - $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

Trig Sub -
1. Write down sub, calculate dx and specify θ
2. Sub expression and dx into ∫ and simplify
3. Integrate - keep in mind θ restrictions
4. Draw a reference Δ to reverse the sub to original x



Impropper Integrals - 2 types

1. integrals with infinate limits.

1. If $f(x)$ is continuous on $[a, \infty)$, then $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $(-\infty, b]$, then $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$ where c is any real number.

- if the lim DNE or is ∞ → diverges
- if it has a value → converges.

2. functions that become infinate at a point

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then $$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then $$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Sequences - a list of numbers given in a specific order. Each a_n represents the n-th term. n is called an index. We sometimes give a sequence using its rules. $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$.

a_n converges if $\lim_{n\to\infty} a_n = L$.

- if it does give a number L; it converges and converges to that point.

if $\lim_{n\to\infty} = \pm\infty$; a_n's get larger and larger or smaller and smaller without a bound.

- the sequence might also oscilate; goes from ex. $-1, 1, -1, 1, -1\ldots$ this means it just diverges.

- Sandwich Therem; if $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty}$ of a_n and c_n converges to the same limit, b_n also must converge to that.

Unit 3: 10.2 - 10.6

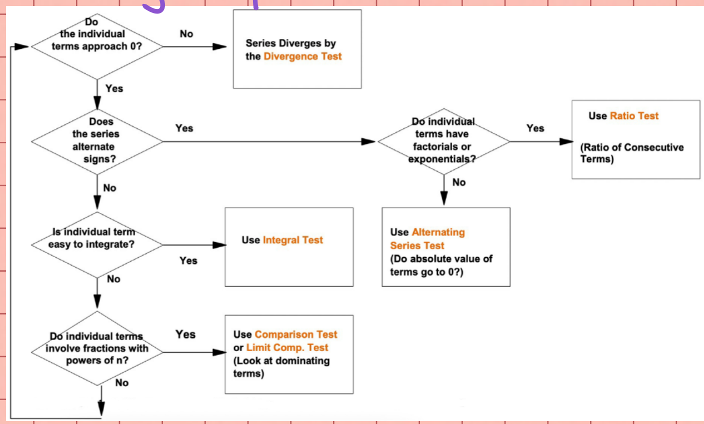
Infinite Series - the sum of a ∞ sequence

$a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_n + \ldots$

Partial Sums - $S_n = \sum_{n=1}^{\infty} a_n$

- the partial sums form a sequence
- if the sequence of partial sums converges to $\lim_{n\to\infty} a_n = S$; the series converges and the sum is S. If the lim DNE the series diverges and if it lim is ∞ it diverges to ∞

Choosing Comparison Tests



nth Term Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition(s) of Convergence: None. Cant show convergence

Condition(s) of Divergence: $\lim_{n\to\infty} a_n \neq 0$

Geometric Series

Series: $\sum_{n=0}^{\infty} ar^n$

Condition of Convergence: $|r| < 1$

Sum: $S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$

Condition of Divergence: $|r| \ge 1$

P-Series Test

Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Condition of Convergence: $p > 1$

Condition of Divergence: $p \le 1$

Telescoping Series

Series: $\sum_{n=1}^{\infty} (a_{n+1} - a_n)$

Condition of Convergence: $\lim_{n\to\infty} a_n = L$

Condition of Divergence: None

Alternating Series

Series: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

Condition of Convergence: $0 < a_{n+1} \le a_n$, $\lim_{n\to\infty} a_n = 0$ or if $\sum_{n=0}^{\infty} |a_n|$ is convergent

Condition of Divergence: cant show divergence

Remainder: $|R_n| \le a_{n+1}$

Integral Test

Series: $\sum_{n=1}^{\infty} a_n$ when $a_n = f(n) \ge 0$ and $f(n)$ is continuous, positive and decreasing

Condition of Convergence: $\int_1^{\infty} f(x)dx$ converges → then Σ converg

Condition of Divergence: $\int_1^{\infty} f(x)dx$ diverges → then Σ diverges

* Remainder: $0 < R_N \le \int_N^{\infty} f(x)dx$

Direct Comparison

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence: $0 < a_n \le b_n$ and $\sum_{n=0}^{\infty} b_n$ is absolutely convergent

Condition of Divergence: $0 < b_n \le a_n$ and $\sum_{n=0}^{\infty} b_n$ diverges

Limit Comparison

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence: $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=0}^{\infty} b_n$ converges

Condition of Divergence: $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=0}^{\infty} b_n$ diverges

Ratio Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

Condition of Divergence: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

* Test *inconclusive* if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Root Test

Series: $\sum_{n=1}^{\infty} a_n$

Condition of Convergence: $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

Condition of Divergence: $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

* Test *inconclusive* if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Unit 4: 10.7 - 10.10, 11.1, 11.2

Power Series; $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1(x-c) + a_2(x-c)$

- Variable = x; a = coefficent; n = power; c = center

Convergence; converge on an interval

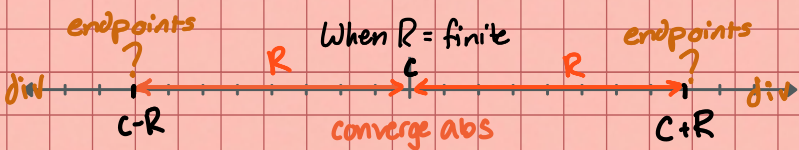
- use ratio test; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$; $\rho<1$: con abs; $\rho>1$: div; $\rho=1$: inconclusive

Outcome 1: F(x) converges abs for all values of x

Outcome 2: there is a radius of convergence R

↳ F(x) converges abs if $|x-c| < R$

↳ F(x) diverges if $|x-c| > R$



Steps For Convergence -

1) Find a_n
2) $\left|\frac{a_{n+1}}{a_n}\right|$
3) Rearange and Simplify
4) Take lim w/ x
5) Use ratio test condition
6) Solve for x to get interval
7) check endpoints

Rewriting a Power Series expansion

Find expansion w/ c=0 for $f(x) = \frac{1}{2+x^2}$

$f(x) = \frac{1}{2} \cdot \frac{1}{1+\frac{1}{2}x^2} \to \frac{1}{2} \cdot \frac{1}{1-(-\frac{1}{2}x^2)}$

$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}x^2\right)^n = \frac{(-1)^n}{2^{n+1}} x^{2n}$

Differentiation & Integration

- $f'(x)=\sum_{n=1}^{\infty} na_n(x-b)^{n-1}$
- $\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$
- term by term

R.O.C is still R

Taylor Polynomials

- Given fn, can you come up w/ a power series for it.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Series

- is infinitley differntiable @ x=c then the Taylor Series for f(x) centered.

$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$

$= \sum_{n=0}^{\infty} \frac{f^{(n)}c}{n!}(x-c)^n$

n^{th} Power Of Series

$\rightarrow C_n = \frac{f^{(n)}(a)}{n!}$

Maclaurin Series

- series centered @ c=0

- $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$

Parametric Curves

- Curves that are traced out by a particle, denoted by 2 seperate equations dependent on t
- To graph; create a table of values and plug in t values for x & y equations sepratley
- You can combine equations to eliminate the parameter t
- We can use $x=\cos t$ and $y=\sin t$ on the unit circle
- For cycloids $x=at-a\cos t$ and $y=a+a\sin t$
- For the bead thing sliding on a string. $T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1+(dy/dx)^2}{2gy}}\,dx$
- A brachistochrone = shortest-time curve which makes the cycloid a tautochrone (same-time curve)

First Derivative

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Area

$$\int_{\alpha}^{\beta} g(t)f'(t)dt$$

Surface Area

around x axis

$$SA(x) = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Arc Length

$$\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt$$

Surface Area

around y axis

$$SA(y) = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Unit '5' - 11.3 - 11.5 & Complex Numbers

Polar Cordinates

- $P(r,\theta)$

r = radius ; θ = angle from initial ray (usually $\theta = 0, 2\pi$)

Polar and Cartesian Cords

- can use to convert between eachother.

$x = r\cos\theta$ $\quad y = r\sin\theta$

$r^2 = x^2+y^2$ $\quad \tan\theta = y/x$

Graphing

- θ = ind var ; r = dep var

$r = f(\theta)$; plug in θ; plot as (r,θ)

- use symmetry

Slope of Polar Equation

- $$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Complex Numbers

- $z = x+iy$ (rectangular form); $Re(z) = x$ (x axis) $\quad Im(z) = y$ (y axis)

$i^2 = -1$

Addition/Subtraction/Scalar Multiplication

; think of i as a variable.

- $(x_1+iy_1)+(x_2+iy_2) = x_1+iy_1+x_2+iy_2$
- $(x_1+iy_1)-(x_2+iy_2) = x_1+iy_1-x_2-iy_2$
- $r(x+iy) = rx+iry$

Multiplication

- basiclly just foil and plug in -1 for i^2

- $(x_1+iy_1)\cdot(x_2+iy_2) = x_1\cdot x_2 + x_1\cdot(iy_2) + (iy_1)\cdot x_2 + (iy_1)\cdot(iy_2) +$
$x_1\cdot x_2 + i(x_1\cdot y_2) + i(x_2\cdot y_1) + i^2(y_1\cdot y_2) = (x_1\cdot x_2 - y_1\cdot y_2) + i(x_1\cdot y_2 + x_2\cdot y_1)$

Complex Conjugate / z bar

- $\overline{z} = \overline{x+iy} = x-iy$

Modules

- $|z| = \sqrt{x^2+y^2} = \sqrt{(Re(z))^2+(Im(z))^2}$

↳ $\overline{\overline{z}} = z$ $\quad z+\overline{z} = 2$ $\quad \overline{x_1+x_2} = \overline{x_1}+\overline{x_2}$ $\quad \overline{x_1\cdot x_2} = \overline{x_1}\cdot\overline{x_2}$

$z\cdot\overline{z} = |z|^2$ $\quad |z_1+z_2| = |z_1|\cdot|z_2|$ *conditonaly

Polar Form of z

- $r(\cos\theta + i\sin\theta)$ or $r=|z|$

Reciprocal

- $$\frac{1}{z} = \frac{Re(z)}{|z|^2} - i\frac{Im(z)}{|z|^2} = \frac{Re(z)}{Re(z)^2+Im(z)^2} - i\frac{Im(z)}{Re(z)^2+Im(z)^2}$$

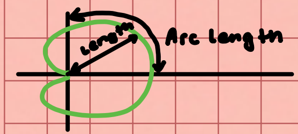
Quotient

- $$\frac{z_1}{z_2} = z_1\cdot z_2^{-1} = z_1\cdot\frac{\overline{z_2}}{|z_2|^2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} - i\frac{x_1y_2-x_2y_1}{x_2^2+y_2^2}$$

Area - $\int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta$ - when more than 1 region; sub 2 integrals

Arc Length - $\int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

Length - $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$



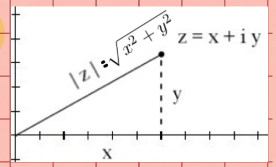
$e^{i\pi} = -1$

*rectangular = polar = exponential

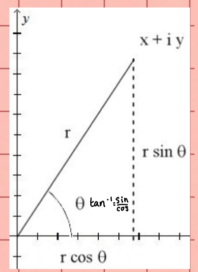
Eulers Formula - $e^{i\theta} = \cos\theta + i\sin\theta$

Exponential Form - $z = re^{i\theta}$ or $z = |z|e^{i\theta}$

Rectangular Cordinate -



Polar Cordinates



Products, Quotients and Powers of CN in polar & exponential

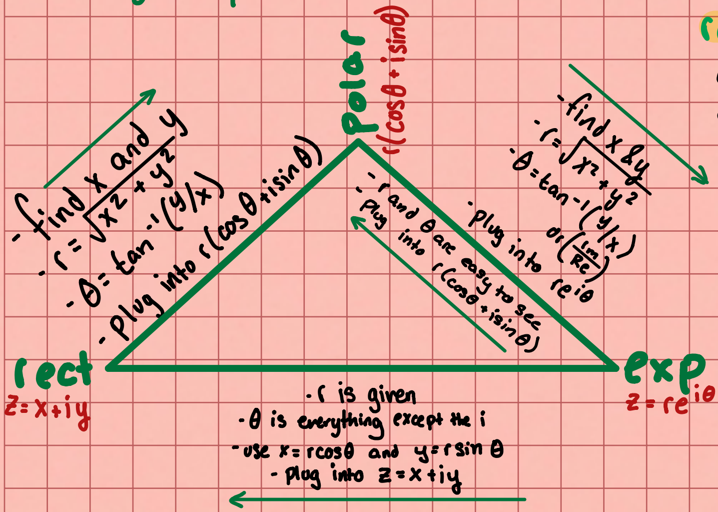
- $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right) = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
- $(re^{i\theta})^k = r^k e^{ik\theta}$

roots - if you take an angle and add 2π to it; it wont change. You can take your original point add 2π / the number thats the power.

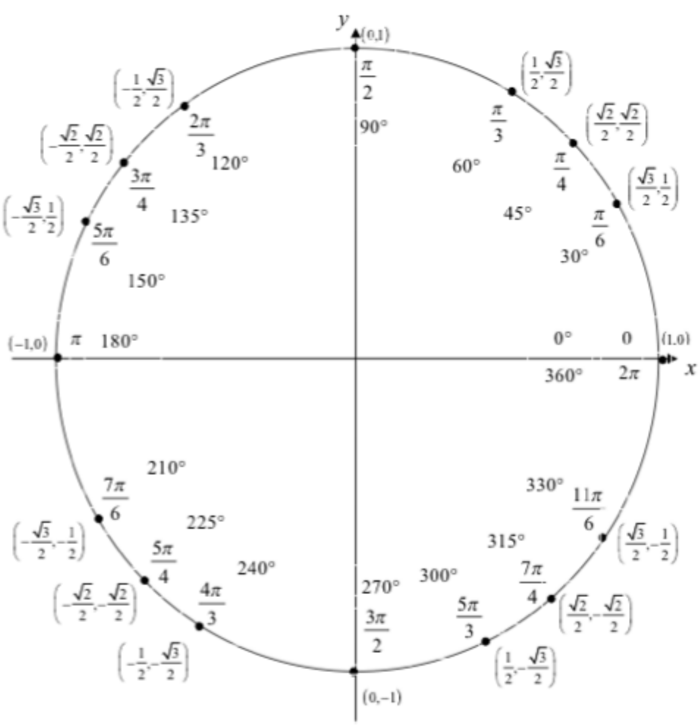
Ex. 4th roots of $4e^{2\pi i/3}$

$(4e^{\frac{2\pi i}{3}})^{1/4} = \sqrt{2}\, e^{\frac{\pi}{6} i}$

$(\sqrt{2}\, e^{i(\frac{\pi}{6}+\frac{2\pi}{4})})^4 = 4e^{i(\frac{2\pi}{3}+2\pi)}$ ← can keep adding sections of 2π



Unit Circle



For any ordered pair on the unit circle (x,y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$ $\quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Inverse Trig Functions

Definition

$y = \sin^{-1} x$ is equivalent to $x = \sin y$

$y = \cos^{-1} x$ is equivalent to $x = \cos y$

$y = \tan^{-1} x$ is equivalent to $x = \tan y$

Inverse Properties

$\cos(\cos^{-1}(x)) = x$ $\quad \cos^{-1}(\cos(\theta)) = \theta$

$\sin(\sin^{-1}(x)) = x$ $\quad \sin^{-1}(\sin(\theta)) = \theta$

$\tan(\tan^{-1}(x)) = x$ $\quad \tan^{-1}(\tan(\theta)) = \theta$

Domain and Range

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

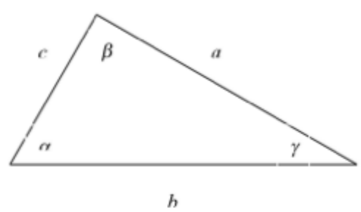
Alternate Notation

$\sin^{-1} x = \arcsin x$

$\cos^{-1} x = \arccos x$

$\tan^{-1} x = \arctan x$

Law of Sines, Cosines and Tangents



Law of Sines

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Mollweide's Formula

$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$

Law of Tangents

$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$

$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$

$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$

Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2-b^2x^2} \Rightarrow x=\frac{a}{b}\sin\theta$ | $\sqrt{b^2x^2-a^2} \Rightarrow x=\frac{a}{b}\sec\theta$ | $\sqrt{a^2+b^2x^2} \Rightarrow x=\frac{a}{b}\tan\theta$

$\cos^2\theta=1-\sin^2\theta$ | $\tan^2\theta=\sec^2\theta-1$ | $\sec^2\theta=1+\tan^2\theta$

Ex. $\int \frac{16}{x^2\sqrt{4-9x^2}}dx$

$x=\frac{2}{3}\sin\theta \Rightarrow dx=\frac{2}{3}\cos\theta\, d\theta$

$\sqrt{4-9x^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2|\cos\theta|$

Recall $\sqrt{x^2}=|x|$. Because we have an indefinite integral we'll assume positive and drop absolute value bars. If we had a definite integral we'd need to compute θ's and remove absolute value bars based on that and,

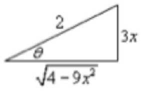
$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0\end{cases}$$

In this case we have $\sqrt{4-9x^2}=2\cos\theta$.

$\int \frac{16}{\frac{4}{9}\sin^2\theta(2\cos\theta)}\left(\frac{2}{3}\cos\theta\right)d\theta=\int\frac{12}{\sin^2\theta}d\theta$

$=\int 12\csc^2 d\theta=-12\cot\theta+c$

Use Right Triangle Trig to go back to x's. From substitution we have $\sin\theta=\frac{3x}{2}$ so,



From this we see that $\cot\theta=\frac{\sqrt{4-9x^2}}{3x}$. So,

$\int\frac{16}{x^2\sqrt{4-9x^2}}dx=-\frac{4\sqrt{4-9x^2}}{x}+c$

Partial Fractions : If integrating $\int\frac{P(x)}{Q(x)}dx$ where the degree of $P(x)$ is smaller than the degree of $Q(x)$. Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor in $Q(x)$	Term in P.F.D	Factor in $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Ex. $\int\frac{7x^2+13x}{(x-1)(x^2+4)}dx$

$\int\frac{7x^2+13x}{(x-1)(x^2+4)}dx=\int\frac{4}{x-1}+\frac{3x+16}{x^2+4}dx$

$=\int\frac{4}{x-1}+\frac{3x}{x^2+4}+\frac{16}{x^2+4}dx$

$=4\ln|x-1|+\frac{3}{2}\ln(x^2+4)+8\tan^{-1}\left(\frac{x}{2}\right)$

Here is partial fraction form and recombined.

$\frac{7x^2+13x}{(x-1)(x^2+4)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{A(x^2+4)+(Bx+C)(x-1)}{(x-1)(x^2+4)}$

Set numerators equal and collect like terms.

$7x^2+13x=(A+B)x^2+(C-B)x+4A-C$

Set coefficients equal to get a system and solve to get constants.

$A+B=7$ $C-B=13$ $4A-C=0$

$A=4$ $B=3$ $C=16$

An alternate method that *sometimes* works to find constants. Start with setting numerators equal in previous example : $7x^2+13x=A(x^2+4)+(Bx+C)(x-1)$. Chose *nice* values of x and plug in. For example if $x=1$ we get $20=5A$ which gives $A=4$. This won't always work easily.

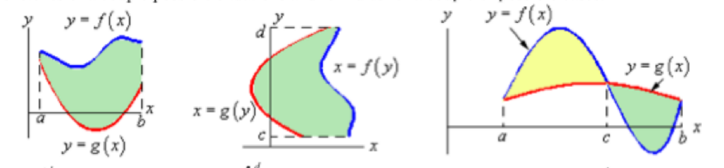
Applications of Integrals

Net Area : $\int_a^b f(x)dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.

Area Between Curves : The general formulas for the two main cases for each are,

$y=f(x) \Rightarrow A=\int_a^b[\text{upper function}]-[\text{lower function}]dx$ & $x=f(y) \Rightarrow A=\int_c^d[\text{right function}]-[\text{left function}]dy$

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.

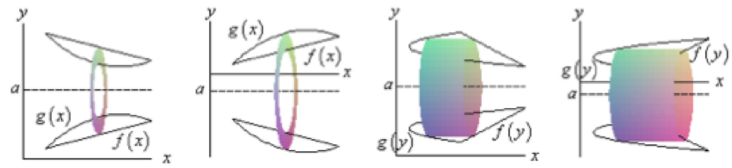


$A=\int_a^b f(x)-g(x)dx$ $A=\int_c^d f(y)-g(y)dy$ $A=\int_a^c f(x)-g(x)dx+\int_c^b g(x)-f(x)dx$

Volumes of Revolution : The two main formulas are $V=\int A(x)dx$ and $V=\int A(y)dy$. Here is some general information about each method of computing and some examples.

Rings	Cylinders
$A=\pi\left((\text{outer radius})^2-(\text{inner radius})^2\right)$	$A=2\pi(\text{radius})(\text{width / height})$
Limits: x/y of right/bot ring to x/y of left/top ring	Limits : x/y of inner cyl. to x/y of outer cyl.
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx. Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy. Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

Ex. Axis : $y=a>0$ **Ex.** Axis : $y=a\leq 0$ **Ex.** Axis : $y=a>0$ **Ex.** Axis : $y=a\leq 0$



Ex. Axis : $y=a>0$	Ex. Axis : $y=a\leq 0$	Ex. Axis : $y=a>0$	Ex. Axis : $y=a\leq 0$
outer radius : $a-f(x)$	outer radius: $\lvert a\rvert+g(x)$	radius : $a-y$	radius : $\lvert a\rvert+y$
inner radius : $a-g(x)$	inner radius: $\lvert a\rvert+f(x)$	width : $f(y)-g(y)$	width : $f(y)-g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y=a\leq 0$ case with $a=0$. For vertical axis of rotation ($x=a>0$ and $x=a\leq 0$) interchange x and y to get appropriate formulas.

Work : If a force of $F(x)$ moves an object in $a\leq x\leq b$, the work done is $W=\int_a^b F(x)dx$

Average Function Value : The average value of $f(x)$ on $a\leq x\leq b$ is $f_{avg}=\frac{1}{b-a}\int_a^b f(x)dx$

Arc Length Surface Area : Note that this is often a Calc II topic. The three basic formulas are,

$L=\int_a^b ds$ $SA=\int_a^b 2\pi y\, ds$ (rotate about x-axis) $SA=\int_a^b 2\pi x\, ds$ (rotate about y-axis)

where ds is dependent upon the form of the function being worked with as follows.

$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$ if $y=f(x)$, $a\leq x\leq b$ $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$ if $x=f(t), y=g(t)$, $a\leq t\leq b$

$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$ if $x=f(y)$, $a\leq y\leq b$ $ds=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta$ if $r=f(\theta)$, $a\leq\theta\leq b$

With surface area you *may* have to substitute in for the x or y depending on your choice of ds to match the differential in the ds. With parametric and polar you will always need to substitute.

Improper Integral

An improper integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called convergent if the limit exists and has a finite value and divergent if the limit doesn't exist or has infinite value. This is typically a Calc II topic.

Infinite Limit

1. $\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx$ 2. $\int_{-\infty}^b f(x)dx=\lim_{t\to-\infty}\int_t^b f(x)dx$

3. $\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^c f(x)dx+\int_c^\infty f(x)dx$ provided BOTH integrals are convergent.

Discontinuous Integrand

1. Discont. at a: $\int_a^b f(x)dx=\lim_{t\to a^+}\int_t^b f(x)dx$ 2. Discont. at b : $\int_a^b f(x)dx=\lim_{t\to b^-}\int_a^t f(x)dx$

3. Discontinuity at $a<c<b$: $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$ provided both are convergent.

Comparison Test for Improper Integrals : If $f(x)\geq g(x)\geq 0$ on $[a,\infty)$ then,

1. If $\int_a^\infty f(x)dx$ conv. then $\int_a^\infty g(x)dx$ conv. 2. If $\int_a^\infty g(x)dx$ divg. then $\int_a^\infty f(x)dx$ divg.

Useful fact : If $a>0$ then $\int_a^\infty\frac{1}{x^p}dx$ converges if $p>1$ and diverges for $p\leq 1$.

Approximating Definite Integrals

For given integral $\int_a^b f(x)dx$ and a n (must be even for Simpson's Rule) define $\Delta x=\frac{b-a}{n}$ and divide $[a,b]$ into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, … , $[x_{n-1},x_n]$ with $x_0=a$ and $x_n=b$ then,

Midpoint Rule : $\int_a^b f(x)dx\approx\Delta x\left[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right]$, x_i^* is midpoint $[x_{i-1},x_i]$

Trapezoid Rule : $\int_a^b f(x)dx\approx\frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$

Simpson's Rule : $\int_a^b f(x)dx\approx\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$

Products and (some) Quotients of Trig Functions

$\int\sin^n x\cos^m x\,dx$

1. **If n is odd.** Strip one sine out and convert the remaining sines to cosines using $\sin^2 x=1-\cos^2 x$, then use the substitution $u=\cos x$
2. **If m is odd.** Strip one cosine out and convert the remaining cosines to sines using $\cos^2 x=1-\sin^2 x$, then use the substitution $u=\sin x$
3. **If n and m are both odd.** Use either 1. or 2.
4. **If n and m are both even.** Use double angle formula for sine and/or half angle formulas to reduce the integral into a form that can be integrated.

$\int\tan^n x\sec^m x\,dx$

1. **If n is odd.** Strip one tangent and one secant out and convert the remaining tangents to secants using $\tan^2 x=\sec^2 x-1$, then use the substitution $u=\sec x$
2. **If m is even.** Strip two secants out and convert the remaining secants to tangents using $\sec^2 x=1+\tan^2 x$, then use the substitution $u=\tan x$
3. **If n is odd and m is even.** Use either 1. or 2.
4. **If n is even and m is odd.** Each integral will be dealt with differently.

The integral contains...	Corresponding Substitution	Useful Identity
a^2-x^2	$x=a\sin\theta$, $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, for $\lvert x\rvert\leq a$	$a^2-a^2\sin^2\theta=a^2\cos^2\theta$
a^2+x^2	$x=a\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$a^2+a^2\tan^2\theta=a^2\sec^2\theta$
x^2-a^2	$x=a\sec\theta$, $\begin{cases}0\leq\theta<\frac{\pi}{2}, \text{ for } x\geq a\\ \frac{\pi}{2}<\theta\leq\pi, \text{ for } x\leq -a\end{cases}$	$a^2\sec^2\theta-a^2=a^2\tan^2\theta$

Function	Maclaurin series expansion	Sigma Notation
$\frac{1}{1-x}$	$1+x+x^2+x^3+x^4+\cdots$	$\sum_{n=0}^{\infty}x^n$
$\cos x$	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$
e^x	$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^2\ldots$	$\sum_{n=0}^{\infty}\frac{1}{n!}x^n$
$\ln(1+x)$	$x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}x^n$
$\sin(x)$	$x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

Definitions

Basic Series

Infinite Sequence: $\langle s_n \rangle$

Limit/Convergence of a Sequence: $\lim_{n\to\infty} s_n = L$

Infinite Serie: (Partial sums) $S_n = \sum s_n = s_1 + s_2 + \cdots + s_n + \cdots$

Geometric Serie:

$$\sum_{k=1}^{n} ar^{k-1} = S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Positive Series

Positive Serie: If all the terms s_n are positive.

Integral Test: If $f(n) = s_n$, continuous, positive, decreasing: $\sum s_n$ converges $\iff \int_1^\infty f(x)dx$ converges.

Comparison Test: $\sum a_n$ and $\sum b_n$ where $a_k < b_k \quad (\forall k \geq m)$
1. If $\sum b_n$ converges, so does $\sum a_n$
2. If $\sum a_n$ diverges, so does $\sum b_n$

Limit Comparison Test: $\sum a_n$ and $\sum b_n$ such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, $\sum a_n$ converges $\iff \sum b_n$ converges.

Convergence

Alternating Serie:

$$\sum (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

Absolute Convergence: If $\sum |s_n|$ is convergent.

Conditional Convergence: If $\sum s_n$ is convergent but *not* absolutely convergent.

Ratio Test: If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| =$
- < 1: absolutely convergent
- 1: (no conclusion)
- > 1 or $+\infty$: diverges

Root Test: If $\lim_{n\to\infty} \sqrt[n]{|s_n|} =$
- < 1: absolutely convergent
- 1: (no conclusion)
- > 1 or $+\infty$: diverges

Uniform Convergence: If $\forall \epsilon > 0, \exists m$ such that for each x and every $n \geq m$, $\quad f_n(x) - f(x) < \epsilon$

Power Series

Power Serie:

$$\sum_{n=0}^{+\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Power Serie About Zero:

$$\sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Taylor Serie

If f a function infinitely differentiable,

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

MacLaurin Serie

If f a function infinitely differentiable,

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Taylor's Formula with Remainder

$\exists x^*$ between c and x such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1}$$

Applications

Application: Showing Function/Taylor-Series Equivalence

$$\lim_{n\to+\infty} R_n(x) = 0$$

Application: Approximating Functions or Integrals

$$R_n(x_0) < K$$

Binomial Serie

$$(1+x)^r = 1 + \sum_{n=1}^{+\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n$$

Common Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots +$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1}\frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

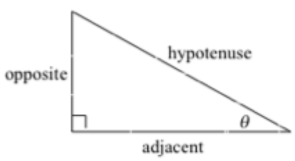
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Trig Cheat Sheet

Definition of the Trig Functions

Right triangle definition

For this definition we assume that $0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.



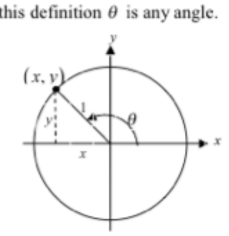
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ $\quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

Unit circle definition

For this definition θ is any angle.



$\sin\theta = \frac{y}{1} = y \quad \csc\theta = \frac{1}{y}$

$\cos\theta = \frac{x}{1} = x \quad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin\theta$, θ can be any angle

$\cos\theta$, θ can be any angle

$\tan\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm1, \pm2, \ldots$

$\csc\theta$, $\theta \neq n\pi$, $n = 0, \pm1, \pm2, \ldots$

$\sec\theta$, $\theta \neq \left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm1, \pm2, \ldots$

$\cot\theta$, $\theta \neq n\pi$, $n = 0, \pm1, \pm2, \ldots$

Range

The range is all possible values to get out of the function.

$-1 \leq \sin\theta \leq 1 \quad \csc\theta \geq 1$ and $\csc\theta \leq -1$

$-1 \leq \cos\theta \leq 1 \quad \sec\theta \geq 1$ and $\sec\theta \leq -1$

$-\infty < \tan\theta < \infty \quad -\infty < \cot\theta < \infty$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$\sin(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$

$\cos(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$

$\tan(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$

$\csc(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$

$\sec(\omega\theta) \rightarrow T = \frac{2\pi}{\omega}$

$\cot(\omega\theta) \rightarrow T = \frac{\pi}{\omega}$

Formulas and Identities

Tangent and Cotangent Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$

Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta} \quad \sin\theta = \frac{1}{\csc\theta}$

$\sec\theta = \frac{1}{\cos\theta} \quad \cos\theta = \frac{1}{\sec\theta}$

$\cot\theta = \frac{1}{\tan\theta} \quad \tan\theta = \frac{1}{\cot\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Even/Odd Formulas

$\sin(-\theta) = -\sin\theta \quad \csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta \quad \sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta \quad \cot(-\theta) = -\cot\theta$

Periodic Formulas

If n is an integer.

$\sin(\theta + 2\pi n) = \sin\theta \quad \csc(\theta + 2\pi n) = \csc\theta$

$\cos(\theta + 2\pi n) = \cos\theta \quad \sec(\theta + 2\pi n) = \sec\theta$

$\tan(\theta + \pi n) = \tan\theta \quad \cot(\theta + \pi n) = \cot\theta$

Double Angle Formulas

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180}$ and $x = \frac{180t}{\pi}$

Half Angle Formulas (alternate form)

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \quad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$

$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$

Sum and Difference Formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Product to Sum Formulas

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

Sum to Product Formulas

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Cofunction Formulas

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \quad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

SERIES			WHEN IS VALID/TRUE
$\frac{1}{1-x}$	$=$	$1 + x + x^2 + x^3 + x^4 + \ldots$	NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF x AS r
	$=$	$\sum_{n=0}^{\infty} x^n$	$x \in (-1, 1)$
e^x	$=$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	$=$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	NOTE $y = \cos x$ IS AN EVEN FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY EVEN POWERS.
	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	$=$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$	NOTE $y = \sin x$ IS AN ODD FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY ODD POWERS.
	$=$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\ln(1+x)$	$=$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$	QUESTION: IS $y = \ln(1+x)$ EVEN, ODD, OR NEITHER?
	$=$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \stackrel{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x \in (-1, 1]$
$\tan^{-1} x$	$=$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$	QUESTION: IS $y = \arctan(x)$ EVEN, ODD, OR NEITHER?
	$=$	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x \in [-1, 1]$

Big Questions 3. For what values of x does the power (a.k.a. Taylor) series

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \qquad (1)$$

converge (usually the Root or Ratio test helps us out with this question). If the power/Taylor series in formula (1) does indeed converge at a point x, does the series converge to what we would want it to converge to, i.e., does

$$f(x) = P_\infty(x) \ ? \qquad (2)$$

Question (2) is going to take some thought.

Definition 4. The *N^{th}-order Remainder term* for $y = f(x)$ at x_0 is:

$$R_N(x) \stackrel{\text{def}}{=} f(x) - P_N(x)$$

where $y = P_N(x)$ is the N^{th}-order Taylor polynomial for $y = f(x)$ at x_0.

So

$$f(x) = P_N(x) + R_N(x) \qquad (3)$$

that is

$$f(x) \approx P_N(x) \qquad \text{within an error of} \quad R_N(x) .$$

We often think of all this as:

$$f(x) \approx \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \qquad \hookleftarrow \text{a finite sum, the sum stops at } N .$$

We would LIKE TO HAVE THAT

$$f(x) \stackrel{??}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \qquad \hookleftarrow \text{the sum keeps on going and going} .$$

In other notation:

$$f(x) \approx P_N(x) \qquad \text{and the question is} \qquad f(x) \stackrel{??}{=} P_\infty(x)$$

where $y = P_\infty(x)$ is the Taylor series of $y = f(x)$ at x_0.

Well, let's think about what needs to be for $f(x) \stackrel{??}{=} P_\infty(x)$, i.e., for f to equal to its Taylor series.

Notice 5. Taking the $\lim_{N\to\infty}$ of both sides in equation (3), we see that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \qquad \hookleftarrow \text{the sum keeps on going and going} .$$

if and only if

$$\lim_{N\to\infty} R_N(x) = 0 .$$

Recall 6. $\lim_{N\to\infty} R_N(x) = 0$ if and only if $\lim_{N\to\infty} |R_N(x)| = 0$.

So 7. If

$$\lim_{N\to\infty} |R_N(x)| = 0 \qquad (4)$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n .$$

So we basically want to show that (4) holds true.

Complex #

Definition

$z = x + iy$

$i = \sqrt{-1};\ x = \mathrm{Re}(z);\ y = \mathrm{Im}(z)$

Algebra

$z_1 = x_1 + iy_1;\ z_2 = x_2 + iy_2$

$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - y_2 x_1)}{x_2^2 + y_2^2}$$

Modulus: |z|

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

Properties

$|z| = |-z| = |\bar{z}| = |-\bar{z}|$

$|z| = \pm\mathrm{Re}(z) \Leftrightarrow \mathrm{Im}(z) = 0$

$|z| = \pm\mathrm{Im}(z) \Leftrightarrow \mathrm{Re}(z) = 0$

$|z_1 z_2| = |z_1||z_2|$

$|z^n| = |z|^n$

$|z_1/z_2| = |z_1|/|z_2|$

$|z_1 + z_2| \le |z_1| + |z_2|$

Equality holds when $0, z_1$, and z_2 are collinear and z_1 and z_2 are on the same side of 0

$|z_1 - z_2| \ge ||z_1| - |z_2||$

Equality holds when $0, z_1$, and z_2 are collinear and z_1 and z_2 are on the same side of 0

Conjugate: $\bar{z}$

$z = x + iy \Rightarrow \bar{z} = x - iy$

Properties

$\bar{\bar{z}} = z$

$\overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}$

$\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$

$\overline{(z^n)} = (\bar{z})^n$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

$|\bar{z}| = |z|$

$\arg(\bar{z}) = \underset{k \in Z}{2k\pi} - \arg(z)$

$\bar{z} = z \Leftrightarrow \mathrm{Im}(z) = 0$

$\bar{z} = -z \Leftrightarrow \mathrm{Re}(z) = 0$